Indian Statistical Institute, Bangalore
M. Math.
Second Year, Second Semester
Advanced Linear Algebra

Mid-term examination — Date : Feb. 15, 2025
Total Marks: 105 — Time: 3 hours
Maximum marks: 100 — Instructor: B V Rajarama Bhat

(1) Fix a natural number n. (i) Show that

$$F = [F_{jk}]_{0\leq j,k\leq (n-1)},$$

where $F_{jk} = \frac{1}{\sqrt{n}} e^{\frac{2\pi ijk}{n}}$ is a unitary matrix. (ii) The 'circulant matrix' with parameters $a_0, a_1, \ldots, a_{n-1}$ is defined as

$$C = [c_{jk}]_{0\leq j,k\leq (n-1)}$$

with $c_{jk} = a_{j-k(\bmod)(n)}$, where a_j's are any complex numbers. Obtain a spectral decomposition for C. (Hint: Use part (i)) [15]

(2) Suppose P_1, P_2 are orthogonal projections in $M_n(\mathbb{C})$. Show that P_1+P_2 is a projection if and only if they are mutually orthogonal that is, $P_1P_2 = P_2P_1 = 0$. [15].

(3) (i) Let $A \in M_n(\mathbb{R})$. Show that A is doubly stochastic iff $Ax \prec x$ for every $x \in \mathbb{R}$. [15]

(ii) Suppose A is an invertible doubly stochastic matrix such that A^{-1} is also doubly stochastic. Show that A must be a permutation matrix. [15]

(4) Let x, y be vectors in $\mathbb{R}^n$. Show that $x \prec_w y$ (x is weakly majorized by y) iff for every strictly increasing convex function $g : \mathbb{R} \to \mathbb{R}$,

$$\sum_{i=1}^{n} g(x_i) \leq \sum_{i=1}^{n} g(y_i).$$

Please provide all details of the proof. [15]

P.T.O.

(5) Let A, B be two self-adjoint matrices in $M_n(\mathbb{C})$. Show that

$$\sum_{j=1}^{k} \lambda_j^{\downarrow}(A+B) \leq \sum_{j=1}^{k} \lambda_j^{\downarrow}(A) + \sum_{j=1}^{k} \lambda_j^{\downarrow}(B),$$

where $\lambda^{\downarrow}$ denotes eigenvalues arranged in decreasing order. [15]

(6) Define $\tau : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by

$$\tau(X) = \text{ trace } (X).I.$$

Show that τ is completely positive. [15]

(7) Define $\alpha : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ and $\beta : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ by

$$\alpha(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$$

$$\beta(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

(i) Show that α, β are unital quantum channels. (ii) Write down Choi-Kraus decompositions for α, β. (iii) Are they mixed unitary channels? [15]